Regressive Welfare Effects of Housing Bubbles

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Abstract

We analyze the welfare effects of asset bubbles in a model with income inequality and financial friction. We show that a bubble that emerges in the value of housing, a durable asset that is fundamentally useful, has regressive welfare effects on users of the asset. By raising the interest rate on debt and the housing price, the bubble benefits high-income savers, but negatively affects low-income borrowers. The regressive effect is generally absent if the model only considers a standard pure bubble with no fundamental value.

Keywords: rational bubble; housing; credit friction; welfare analysis.
JEL classification: E10; E21; E44.

1 Introduction

Many countries have experienced several episodes of bubble-like booms in asset prices. Examples include the real estate and stock price booms in Japan in the 1980s and South East Asia in the 1990s; the housing price booms in Ireland, Spain, and the U.S. in the 2000s; and the current housing price boom in China (Hunter, 2005, Mian and Sufi, 2014, Fang et al., 2015). In general, when there is a high demand for savings but limited investment outlet, the rates of returns from investment are depressed and real estate investment can serve as a prominent store of value. Thus, a low interest rate environment, as seen in the recent decade, provides a fertile ground for the emergence of asset bubbles, especially in real estate. Given the prevalence of bubble episodes, a central question arises for academics and policymakers: What are the welfare effects of asset bubbles, especially bubbles in real estate?

In this paper, we highlight the nuanced welfare effects of bubbles that are attached to housing. We develop a simple overlapping generations (OLG) model of bubbles with intra-generation heterogeneity and financial friction. As described in section 2 of the paper, households have identical preferences over a perishable consumption good and a durable and perfectly divisible housing asset in fixed supply. Young agents receive endowments, and a fraction of them are *savers*, who are born with high endowments, and the remaining fraction are *borrowers*, who are born with low endowments. Young borrowers, given their low endowment, need to borrow to purchase the desired amount of housing that maximizes their utility. In contrast, young savers, given their high endowment, do not need to borrow and instead would like to save some income for old age. Thus, for savers, housing not only yields utility dividend but also serves as a savings vehicle.

In an economy without financial friction, households can achieve their first best allocations by borrowing and lending in the credit market. However, in the presence of financial friction, such as imperfect contract enforcement, borrowers face a binding credit constraint, modeled as an exogenous limit on borrowers' debt capacity, as in Huggett (1993), Aiyagari (1994), and Eggertsson and Krugman (2012). In equilibrium, the constraint effectively limits how much savers can store their income by investing in the credit market. As we show in section 3, in an economy with high income inequality, there is a shortage of storage for savers, which can lead to an equilibrium interest rate that is below the economy's growth rate. The low interest rate environment in turn facilitates the emergence of asset bubbles.

In section 4, the main part of our paper, we study housing bubbles. In a housing bubble equilibrium, the price of one unit of housing consists of a fundamental component equal to the net present value of the stream of utility dividends, and a bubble component, which grows at the interest rate. The housing bubble causes two macroeconomic effects: it raises the equilibrium interest rate and it raises the equilibrium housing price (relative to the bubble-less benchmark).

We then show that the housing bubble has opposite effects on borrowers and savers. On the one hand, the housing bubble increases the return from real estate investment for high-income savers, who demand storage of value, and hence increases their welfare (relative to the bubble-less benchmark). On the other hand, by raising the interest rate on debt and raising the housing price, the bubble *reduces* the purchasing power and hence the welfare of low-income borrowers, who in equilibrium have a relatively high marginal utility from housing. By positively affecting high-income savers and negatively affecting low-income borrowers, the housing bubble thus has *regressive welfare effects*. Overall, the results so far imply a feedback loop on inequality: high income inequality depresses the interest rate, thereby facilitating the existence of housing bubbles, which in turn have regressive welfare effects.

In comparison, section 5 shows that the regressive welfare implications are absent if the model consider pure bubbles, which are widely used in the rational bubble literature for their simplicity. A pure bubble is an asset that has no fundamental value,[1] but which is traded at a positive price. The pure bubble provides an additional investment vehicle for savers: besides investing in the credit market and the housing market, savers can invest in the bubble market by purchasing the bubble asset when young and reselling it when old. However, the bubble provides no useful service for borrowers who do not want to save. Thus, unlike the housing bubble equilibrium, the pure bubble equilibrium is characterized by an endogenous *segmentation* in the bubble market, as only savers purchase the bubble asset. We then show that under some conditions, the pure bubble crowds out savers' investment in the housing market and thus lowers the housing price. This crowd-out effect counteracts the effect on the interest rate and improves borrowers' welfare. Under some conditions, the crowd-out effect dominates the interest rate effect, and as a consequence the pure bubble increases borrowers' welfare (relative to the bubble-less benchmark). Under other conditions, the interest rate effect dominates, and the pure bubble reduces borrowers' welfare. Hence, while the pure bubble continues to improve savers' welfare as the housing bubble, it has ambiguous effects on borrowers.

In summary, our paper shows that a housing bubble (or, more generally, a bubble that emerges on a fundamentally useful asset) can have regressive welfare effects, in the sense that it benefits high-income savers and negatively affects low-income borrowers.

Related literature. Our paper is related to the rational bubble literature, which has a long heritage dating back to Samuelson (1958), Diamond (1965), and Tirole (1985). For a survey of this literature, see Miao (2014).[2] Much of the literature has focused on a positive analysis of bubbles. A common theme in this literature is that rational bubbles emerge to reduce some inefficiency in the financial market, such as an aggregate shortage of assets for storage or a credit market imperfection, as in Hirano and Yanagawa (2010), Miao and Wang (2011), Martin and Ventura (2012), and Ikeda and Phan (2015).

By departing from the pure bubble assumption and modeling a bubble as attached to a fundamentally useful durable asset such as housing, our paper is related to Arce and López-Salido (2011), Miao and Wang (2012), Wang and Wen (2012), Hillebrand and Kikuchi (2015),

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[1]Examples include such as tulips, fiat money, or stocks of an unproductive firm. In fact, the literature often uses the boom in the stock prices of many dot com firms in the U.S. in the late 1990s and early 2000s as an example of a pure bubble (see, inter alia, Martin and Ventura, 2012).

[2]There is another bubble literature that focuses on the role of information in coordinating agents' actions to purchase and sell bubbles. See, inter alia, Brunnermeier (2003), Doblas-Madrid (2012), Barlevy (2014), and Doblas-Madrid and Lansing (2014).

Zhao (2015) and Basco (2016). A common theme among Arce and López-Salido (2011), Zhao (2015), and Basco (2016) is that they focus on setups where agents have heterogeneous preferences for housing, and they define a housing bubble as an equilibrium where some agents, who derive no direct utility or use from an asset, purchase the asset purely as a store of value. In contrast, we focus on setups where agents have identical preferences and all derive utility from housing, and we define a housing bubble as an equilibrium where the price of the housing asset has a bubble component. Our model of fundamental bubbles is thus more related to Blanchard and Watson (1982), Wang and Wen (2012), and Hillebrand and Kikuchi (2015). The main difference is our paper's focus on a welfare analysis. A second difference is our emphasis on income inequality.

To the best of our knowledge, among papers that analyze the welfare effects of bubbles, ours is the first to document regressive welfare effects of a housing bubble. Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) show that if there is a positive externality in the accumulation of capital, the emergence of bubbles on an unproductive asset would inefficiently divert resources from investment. Similarly, Hirano et al. (2015) show that oversized bubbles inefficiently crowd out productive investment. On the other hand, Miao et al. (2015) show that bubbles can crowd in too much investment. Caballero and Krishnamurthy (2006) show that bubbles can marginally crowd out domestic savings and cause a shortage of liquid international assets in a small open economy framework. Focusing instead on risk, Ikeda and Phan (2016) show that rational bubbles financed by credit can be excessively risky. The regressive welfare effects that we highlight are complementary to the effects highlighted by these papers.

2 The Model

Consider an endowment economy with overlapping generations of agents who live for two periods. Time is discrete and infinite, with dates denoted by $t = 0, 1, 2, \ldots$. There is a consumption good and a housing asset. The consumption good is perishable and cannot be stored. The housing asset is durable, perfectly divisible and in a fixed unit supply. The consumption good is the numeraire and the market price of a unit of the housing asset is denoted by p_t.

Heterogeneity. Each generation consists of two groups of households, savers and borrowers (or debtors), denoted by $i \in \{s, d\}$, with equal measure of each group normalized to one. Each young household is endowed with e^i of the consumption good, where $e^s > e^d$. Old households receive no endowment. Let $e^s + e^i = e$ be the total endowment of the consumption good.

Preferences. Households derive utility from the housing asset and from the consumption

good, consumed both when young and old. Denote their utility function by $U(h_t^i, c_{t,y}^i, c_{t+1,o}^i)$, where h_t^i denotes the housing and $c_{t,y}^i$ and $c_{t+1,o}^i$ denote consumption in young and old age of a household of type $i \in \{s, d\}$ born in period t. For simplicity, we assume the functional form:

$$U(h_t^i, c_{t,y}^i, c_{t+1,o}^i) = \theta \log(h_t^i) + \log(c_{t,y}^i) + \beta c_{t+1,o}^i,$$

where $\theta > 0$ is a parameter measuring the strength of housing preferences, $\beta \in (0, 1)$ is the discount factor, and we impose the linearity of utility in old age to simplify the portfolio optimization with risky assets.

Credit market and credit friction. Households can borrow and lend to each other via a credit market. Let $1 + r_t$ denote the gross interest rate for debt between period t and $t+1$. As in Bewley (1977), Huggett (1993), Aiyagari (1994), and Eggertsson and Krugman (2012) we model credit friction in the simplest possible way: an agent can commit to repay at most $\bar{d}$ units of the consumption good, where $\bar{d} > 0$ is an exogenous debt limit. This imperfection in the financial market will lead to a constraint on households' ability to borrow, as manifested in the optimization problem below.

A household purchases housing, consumes, and borrows or lends when young, and then sells their housing asset and consumes when old. As in Lorenzoni (2008) and Hillebrand and Kikuchi (2015), we assume a per-unit maintenance cost $\kappa > 0$ on the durable housing asset.[3] Formally, the optimization problem of a young household of type $i \in \{s, d\}$ born in period t consists of choosing housing asset position h_t^i, net financial asset position a_t^i, and old-age consumption c_{t+1}^i to maximize lifetime utility:

$$\max_{h_t^i, c_{t,y}^i, c_{t+1,o}^i, a_t^i} U(h_t^i, c_{t,y}^i, c_{t+1,o}^i) \tag{1}$$

subject to a budget constraint in young age:

$$p_t h_t^i + \frac{1}{1 + r_t} a_t^i + c_{t,y}^i = e^i,$$

a budget constraint in old age (taking into account the maintenance cost of housing):

$$c_{t+1,o}^i = (p_{t+1} - \kappa) h_t^i + a_t^i$$

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[3]This cost prevents the housing price from exploding when the interest rate falls below the growth rate of the economy.

5

a short-selling constraint on the housing asset:

$$h_t^i \geq 0,$$

non-negativity constraints on consumption in both periods of life:

$$c_{t,y}^i, c_{t+1,o}^i \geq 0$$

and a credit constraint:

$$a_t^i \geq -\bar{d}. \tag{2}$$

Finally, to close the model, without loss of generality assume that the old savers own the entire supply of housing in the initial period $t = 0$. We define an equilibrium as follows:

Definition 1. An *equilibrium* consists of allocation $\{h_t^i, c_{t,y}^i, c_{t+1,o}^i, a_t^i\}_{t \geq 0}$ and prices $\{p_t, r_t\}_{t \geq 0}$ such that:

1. Given prices, the allocations solve the optimization problem (1) for all $i \in \{s, d\}$ and $t \geq 0$.

2. The consumption good market clears:

$$c_{t,y}^s + c_{t,y}^d + c_{t+1,o}^s + c_{t+1,o}^d = e - \kappa, \forall t \geq 0;$$

3. The credit market clears:
$$a_t^s + a_t^d = 0, \forall t \geq 0;$$

4. And the housing marker clears:

$$h_t^s + h_t^d = 1, \forall t \geq 0.$$

We will be mainly focusing on analyses of *steady states*, which are equilibria where quantities and prices are time-invariant.

Throughout the paper we assume $\bar{d}$ is sufficiently small so that the credit constraint (2) is always binding in any equilibrium. Then, the credit market clearing condition implies that savers must be lending and hence are not credit constrained. Therefore, both the equilibrium housing price and interest rate are determined by the first order conditions of

savers. In particular, the equilibrium housing price is given by:[4]

$$p_t = \frac{1}{1+r_t}\left(\frac{\theta}{\beta h_t^s} + p_{t+1} - \kappa\right).$$ (3)

Asset pricing equation (3) states that the price of one unit of housing in period t is equal to the sum of the marginal utility from housing, plus the resale value p_{t+1}, net the maintenance cost, and discounted by the gross interest rate $1+r_t$. We now proceed to show that this setup can support multiple equilibria: a bubble-less equilibrium and a housing bubble equilibrium.

3 Bubble-less Equilibrium

We start with the bubble-less benchmark. In this case, the housing price that solves asset pricing equation (3) is simply the net present value of the net dividend stream:[5]

$$p_t = \sum_{j\geq 0} \frac{1}{\Pi_{k=1}^{j}(1+r_{t+k})}\left(\frac{\theta}{\beta h_{t+j}^s} - \kappa\right).$$ (4)

In other words, the housing price is equal to its fundamental value. Borrowers also purchase the housing asset, and their first order conditions yield an equation similar to (3), but with a different stochastic discount factor due to the fact that borrowers are credit-constrained:[6]

$$p_t = \frac{\theta c_{t,y}^d}{h_t^d} + \beta c_{t,y}^d(p_{t+1} - \kappa).$$ (5)

Equations (4) and (5) are the key equations that pin down equilibrium prices. In fact,

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[4]This equation is derived from savers' first order conditions:

$$p_t\frac{\partial U^s}{\partial c_{t,y}^s} = \frac{\theta}{h_t^s} + (p_{t+1} - \kappa)\beta\frac{\partial U^s}{\partial c_{t+1,o}^s}.$$

Because savers are not credit constrained, we have $\frac{\partial U^s}{\partial c_{t,y}^s} = \beta(1+r_t)\frac{\partial U^s}{\partial c_{t+1,o}^s}$, or equivalently, $1/c_{t,y}^s = \beta(1+r_t)$ which yields (3).

[5]In this equation, to guarantee convergence to a finite and positive value, we impose a sufficient condition: $\theta > \frac{\kappa e^d}{e - \frac{2}{\beta}}$. Note that if $e < \frac{2}{\beta}$ this condition is satisfied for any $\theta > 0$.

[6]This equation is attained by taking the borrowers' first order condition on the choice of the housing asset:

$$p_t\frac{\partial U^d}{\partial c_{t,y}^d} = \frac{\theta}{h_t^d} + \frac{\partial U^d}{\partial c_{t+1,o}^d}(p_{t+1} - \kappa).$$

Note that the marginal utility of old age consumption is simply β, while the marginal utility of young age consumption is $\frac{1}{c_{t,y}^d}$. Since borrowers are credit constrained, we cannot substitute between $\frac{\partial U^d}{\partial c_{t+1,o}^d}$ and $\frac{\partial U^d}{\partial c_{t+1,o}^d}$ as we did for the savers first order conditions.

7

as the following lemma shows, given that borrowers are credit-constrained, there is a unique equilibrium where prices satisfy these equations. We call this the *bubble-less equilibrium.* In our model, as there is no capital accumulation, it can be shown that equilibrium quantities and prices are time-invariant. The equilibrium interest rate r_n and housing price p_n (where the subscript n stands for no-bubble) satisfy the steady state versions of (4) and (5):

$$p_n = \frac{1}{r_n}\left(\frac{\theta}{\beta h_n^s} - \kappa\right), \tag{6}$$

$$= \frac{c_{y,n}^d\left(\frac{\theta}{h_n^d} - \kappa\right)}{1 - \beta c_{y,n}^d}. \tag{7}$$

Lemma 1 (Bubble-less equilibrium)**.** *There exists a unique equilibrium where the housing price is equal to the fundamental value, as given by (4). Equilibrium quantities and prices are time-invariant. The interest rate r_n and housing price p_n solve (6) and (7), with borrowers' consumption and asset positions given by:*

$$a_n^d = -\bar{d},$$
$$h_n^d = 1 - h_n^s,$$
$$c_{y,n}^d = e^d - p_n - c_{y,n}^s,$$
$$c_{o,n}^d = (p_n - \kappa)h_n^d - \bar{d},$$

and savers' consumption and asset positions given by:

$$a_n^s = \bar{d},$$
$$h_n^s = \frac{e^s - \bar{d}\frac{1}{1+r_n} - c_{y,n}^s}{p_n},$$
$$c_{y,n}^s = \frac{1}{\beta(1+r_n)},$$
$$c_{o,n}^s = (p_n - \kappa)h_n^s + \bar{d}.$$

Proof. Appendix A.1. □

Furthermore, it is straightforward to show that r_n is increasing in debt limit $\bar{d}$, and decreasing in savers' endowment e^s. Thus, either an increase in credit friction (a lower $\bar{d}$) or an increase in inequality will lower the interest rate. As is well known, a low interest rate environment is a fertile ground for bubbles to arise. This leads us to the next section.

4   Housing bubble

We will construct a housing bubble equilibrium. In this equilibrium, the price of housing contains a bubble component. Following the literature (e.g., Weil, 1987), we allow for a stochastic bubble by assuming that the bubble component can (permanently) collapse to zero with an exogenous i.i.d. probability π. Formally, the housing price will be given by:

$$p_t = f_t + b_t,$$

where $f_t > 0$ is a "fundamental component," which is equal to the net present value of the net dividend stream as in (4):

$$f_t = \sum_{j \geq 0} \frac{1}{\Pi_{k=0}^j (1 + r_{t+k})} \left(\frac{\theta}{\beta h_{t+j}^s} - \kappa \right), \qquad (8)$$

and $b_t > 0$ is a "bubble component" whose expected value grows at the interest rate:

$$b_t = \frac{1 - \pi}{1 + r_t} b_{t+1}. \qquad (9)$$

Given the stationarity of the bubble-less equilibrium, it is straightforward to verify (see the appendix) that after the bubble component collapses, the housing price immediately reverts to the bubble-less price p_n. Hence, asset pricing equation (3) becomes:

$$f_t + b_t = \frac{1}{1 + r_t} \left(\frac{\theta}{\beta h_t^s} - \kappa + (1 - \pi)(f_{t+1} + b_{t+1}) + \pi p_n \right). \qquad (10)$$

Equation (9) implies that in steady state the interest rate must be:

$$1 + r_h = 1 - \pi,$$

where the subscript h denotes the housing-bubble steady state. Substituting into the steady state versions of (8) and (10) implies that the fundamental component is equal to the bubble-less housing price:

$$f = p_n. \qquad (11)$$

As a consequence, *the housing bubble raises the steady state price of housing*, from p_n to $p_n + b$.

The appendix derives the following expression for the steady state bubble:

$$b = \frac{c_{y,h}^d \left(\frac{\theta}{h_h^d} - \kappa \right) - p_n (1 - \beta c_{y,h}^d)}{1 - \beta c_{d,h}^y (1 - \pi)}.$$ (12)

Equation (12) implies that the size of the bubble is increasing (decreasing) in the borrowers' young age consumption (savers' young-age consumption) and decreasing (increasing) in the borrowers' housing allocation (savers' housing allocation). This is because savers are the marginal buyers of housing (and hence the bubble), and the bubble is most attractive to savers when the lending and housing markets are insufficient to store wealth for old age (leading to over-consumption among young savers). Equation (12) also allows us to pin down conditions under which a housing bubble steady state exists, as we now show.

4.1   Existence and characteristics

From (12), we see that the housing bubble component is positive if and only if the expression on the right hand side is positive. Lemma 2 below shows that this is the case if and only if the bubble-less interest rate is smaller than the housing bubble interest rate:

$$1 + r_n < 1 + r_h = 1 - \pi.$$

Intuitively, when savers are sufficiently wealthy and credit markets have sufficient friction, the economy is dynamically inefficient and there is a shortage of storage for savers. A bubble that arises in the value of housing helps reduce the dynamic inefficiency by raising the returns from housing investment for savers. In other words, the housing bubble improves the storage service provided by the durable housing asset. When the bubble bursts, the economy instantly reverts to the bubble-less steady state equilibrium. This is because there is no capital accumulation in our model, and thus there are no choices made by the young in time t that can possibly impact the choices faced by the young in time $t + 1$.

The lemma below summarizes the existence condition and characterizes the allocations and prices of the housing bubble steady state. It also characterizes how the economy converges to this steady state, depending on the initial value of the bubble.[7]

Lemma 2 (Housing Bubble Equilibrium). *Let f and b be as defined in (10) and (12). Then:*

1. *A housing bubble steady state exists if and only if the bubble-less interest rate is low:*
   $1 + r_n < 1 - \pi.$

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[7] The saddle-path stability of the asymptotic bubble equilibrium is similar to that in Tirole (1985).

2. *The housing bubble steady state is characterized as follows. The interest rate is:*

$$1 + r_h = 1 - \pi.$$

The fundamental component of the housing price f is equal to p_n, the bubble component b is positive and given by (12), where borrowers' consumption and asset positions are given by:

$$
\begin{aligned}
a_h^d &= -\bar{d}, \\
h_h^d &= \frac{e^d + \bar{d}\frac{1}{1-\pi} - c_{y,h}^d}{p_n + b}, \\
c_{y,h}^d &= e - (p_n + b) - c_{y,h}^s, \\
c_{o,h}^d &= \begin{cases} (p_n - \kappa)h_h^d - \bar{d} & \text{if the bubble bursts} \\ (p_n + b - \kappa)h_h^d - \bar{d} & \text{if the bubble persists} \end{cases},
\end{aligned}
$$

and savers' consumption and asset positions are given by:

$$
\begin{aligned}
a_h^s &= \bar{d}, \\
h_h^s &= \frac{e^s - \bar{d}\frac{1}{1-\pi} - c_{y,h}^s}{p_n + b}, \\
c_{y,h}^s &= \frac{1}{\beta(1-\pi)}, \\
c_{o,h}^s &= \begin{cases} (p_n - \kappa)h_h^s + \bar{d} & \text{if the bubble bursts} \\ (p_n + b - \kappa)h_h^s + \bar{d} & \text{if the bubble persists} \end{cases}.
\end{aligned}
$$

If the bubble bursts, the economy immediately reverts to the bubble-less equilibrium with housing price p_n and interest rate r_n.

3. *Assume $1 + r_n < 1 - \pi$. If the initial bubble price satisfies $b_0 < b$, then there exists no asymptotic pure bubble equilibrium, as any equilibrium bubble must vanish: $\lim_{t\to\infty} b_t = 0$. If $b_0 > b$, then there exists no pure bubble equilibrium, as any equilibrium bubble must explode: $\lim_{t\to\infty} b_t = \infty$. If $b_0 = b$, then there exists a unique asymptotic pure bubble equilibrium, and this equilibrium coincides with the pure bubble steady state.*

Proof. Appendix A.2. □

4.2 Welfare analysis

We can now address the main question of interest: What are the welfare effects of the housing bubble? We define the welfare of a household of type $i \in \{s, d\}$ in the housing bubble steady state as $U_h^i = U(h_h^i, c_{y,h}^i, c_{o,h}^i)$, where h_h^i, $c_{y,h}^i$ and $c_{o,h}^i$ are given in lemma 2. Similar for the welfare U_n^i in the bubble-less steady state.

The bubble has heterogeneous effects on savers and borrowers. For savers, who want to save for old age and use the housing asset in part as a savings vehicle, the housing bubble improves welfare. This is because the bubble raises the interest rate (from $1 + r_n$ to $1 - \pi$) and the housing price (from p_n to $p_n + b$), hence increasing the return from lending and housing investment. This effect is summarized in the following lemma:

Lemma 3. *The housing bubble improves savers' welfare: $U_h^s > U_n^s$.*

Proof. Appendix A.3. □

In contrast, the housing bubble has an *unambiguously negative* effect on the welfare of borrowers. This is because it increases the interest rate paid on debt, hence reducing the amount that borrowers can borrow when young. At the same time, it increases the price of housing, hence reducing the amount of housing that borrowers purchase and consequently their housing utility. This effect is summarized in the following lemma:

Lemma 4. *The housing bubble reduces borrowers' welfare: $U_h^d < U_n^d$.*

Proof. Appendix A.4. □

The combination of lemmas 3 and 4 yields the main result of our paper:

Proposition 5 (Regressive welfare effects of housing bubble)**.** *The housing bubble improves welfare for savers but reduces welfare for borrowers (relative to the bubble-less steady state).*

This proposition highlights the regressive welfare effects of a housing bubble. The housing bubble improves welfare for savers by providing them with a more efficient way to store value. However, by raising the cost of debt and the price of housing, the bubble reduces borrowers' ability to purchase housing. Therefore, an interesting implication arises on the bi-directional relationship between inequality and bubble: high income inequality depresses the interest rate, facilitating the existence of housing bubbles, which in turn have regressive welfare effects.

5 Pure bubble equilibrium

To appreciate the welfare results established in the previous section, we compare them against the welfare effects of a pure bubble, which is an asset that pays no dividend but has a positive market price. Common interpretations for this type of asset include a positive price on the stock of an unproductive firm, fiat money, or the famous "tulip mania" in the Netherlands. This asset can be useful as a savings instrument. However, unlike housing, the pure bubble asset does *not* give households any direct utility. As a consequence, there will be an endogenous *segmentation* of the pure bubble market: only savers purchase the asset to store income for old age. This leads to another important distinction between the housing asset and the pure bubble asset: it is never optimal for borrowers to go into debt to acquire the pure bubble.

Formally, assume there is an asset in fixed unit supply. It pays no dividend but is traded at price $\tilde{b}_t$ per unit (assuming that the bubble has not collapsed). Given prices, each household of type i chooses its holding $x_t^i \geq 0$ of the bubble asset. Their optimization problem is:

$$\max_{h_t^i, c_{t,y}^i, c_{t+1,o}^i, x_t^i, a_t^i} U(h_t^i, c_{t,y}^i, c_{t+1,o}^i), \tag{13}$$

subject to young and old age budget constraints:

$$
\begin{aligned}
e^s &= p_t h_t^i + a_t^i \frac{1}{1+r_t} + c_{t,y}^i + \tilde{b}_t x_t^i, \\
c_{t+1,o}^i &= \begin{cases} (p_{t+1} - \kappa)h_t^i + a_t^i & \text{if the bubble bursts} \\ (p_{t+1} - \kappa)h_t^i + a_t^i + \tilde{b}_{t+1}x_t^i & \text{if the bubble persists} \end{cases},
\end{aligned}
$$

a credit constraint:

$$a_t^i \geq -\bar{d},$$

non-negativity constraints on consumption in both periods of life:

$$c_{t,y}^i, c_{t+1,o}^i \geq 0,$$

and no short-selling constraints on housing and the bubble asset:

$$x_t^i, h_t^i \geq 0.$$

To close the model, assume that old savers own the entire supply of housing and the bubble in the initial period $t = 0$.

We retain the assumption from section 4 that the bubble is can permanently collapse to zero in each period with an i.i.d. probability π. As with the housing bubble, it is straightforward to show that the economy will immediately revert to the bubble-less steady state if the bubble bursts.

A pure bubble equilibrium is defined as follows:

Definition 2. A *pure bubble equilibrium* consists of allocation $\{h_t^i, c_{t,y}^i, c_{t+1,o}^i, a_t^i, x_t^i\}_{t \geq 0}$ and prices $\{p_t, r_t, \tilde{b}_t\}_{t \geq 0}$ such that:

1. Given prices, the allocations solve the optimization problem (13) for all $i \in \{s, d\}$ and $t \geq 0$.

2. The consumption good market clears:

$$c_{t,y}^s + c_{t,y}^d + c_{t+1,o}^s + c_{t+1,o}^d + \tilde{b}_t = e - \kappa, \forall t \geq 0;$$

3. The credit market clears:
$$a_t^s + a_t^d = 0, \forall t \geq 0;$$

4. The housing market clears:
$$h_t^s + h_t^d = 1, \forall t \geq 0;$$

5. And the bubble market clears:

$$x_t^s + x_t^d = 1. \forall t \geq 0.$$

We focus on *asymptotic* pure bubble equilibria, where the bubble does not vanish, i.e., $\lim_{t \to \infty} \tilde{b}_t > 0$. A pure bubble *steady state* is an asymptotic pure bubble equilibrium where prices and quantities are time-invariant.

5.1 Existence and characteristics

We now formalize the existence and characteristics of a pure bubble equilibrium. The first order conditions of savers imply that the following no-arbitrage condition must hold for the risky bubble:

$$\tilde{b}_t = \frac{1 - \pi}{1 + r_t} \tilde{b}_{t+1}. \tag{14}$$

This equation equates the return from lending in the credit market and the (expected) return from speculating in the bubble market for savers. It is a standard equation stating that in

any pure bubble equilibrium, the expected bubble price must grow at the interest rate. Intuitively, if this condition does not hold, then either the bubble yields a greater expected return than lending (which means savers would never lend in equilibrium) or lending would have a greater expected return than the bubble (which means savers would have no demand for the bubble). It also implies a standard identity that the interest rate in any pure bubble steady state must be:

$$1 + r_p = 1 - \pi,$$

as in the housing bubble case (the subscript p stands for pure-bubble).

The first order conditions of savers and borrowers yield pricing equations similar to the bubble-less equilibrium:

$$
\begin{aligned}
p_t &= \frac{1}{1 + r_t} \left(\frac{\theta}{\beta h_t^s} + p_{t+1} - \kappa \right), \\
&= \frac{\theta c_{t,y}^d}{h_t^d} + \beta c_{t,y}^d (p_{t+1} - \kappa),
\end{aligned}
$$

which equate the price with discounted dividends of housing for savers and borrowers respectively. In steady state, since $1 + r = 1 - \pi$, these equations become:

$$
p_p = \frac{1}{-\pi} \left(\frac{\theta}{\beta h_p^s} - \kappa \right), \tag{15}
$$

$$
= \frac{c_{y,p}^d \left(\frac{\theta}{h_p^d} - \kappa \right)}{1 - \beta c_{y,p}^d}. \tag{16}
$$

Since only savers invest in the pure bubble, we can use (15) and the savers' young-age budget constraint to find an equation that determines the size of the bubble in equilibrium:

$$
\tilde{b} = e^s - \bar{d} \frac{1}{1 - \pi} - c_{y,p}^s - \frac{\theta p_p}{\kappa - \pi \beta p_p}, \tag{17}
$$

which confirms that the amount that savers invest in the bubble asset is simply whatever income they have left over after consuming the optimal amount in young age, lending to borrowers, and purchasing housing to satisfy (discounted) utility demands. The subsequent lemma shows that $\tilde{b} > 0$ if and only if $1 + r_n < 1 - \pi$, as in the case of the housing bubble. As in lemma 2, it also characterizes the equilibrium allocations and prices, and the convergence to the asymptotic equilibrium.

Lemma 6 (Pure Bubble Equilibrium)**.**

  1. *A pure bubble steady state exists if and only if $1 + r_n < 1 - \pi$.*

2. *The pure bubble steady state is characterized as follows. There is a segmentation in the bubble market, as only savers have demand for the bubble asset:*

$$x^d = 0, x^s = 1.$$

The interest rate is:

$$1 + r_p = 1 - \pi.$$

The housing price p_p and bubble price $\tilde{b}$ are solve (16) and (17), where borrowers' consumption and asset positions are given by:

$$
\begin{aligned}
a_p^d &= -\bar{d}, \\
h_p^d &= \frac{e^d + \bar{d}\frac{1}{1-\pi} - c_{y,p}^d}{p_p}, \\
c_{y,p}^d &= e - p_p - b - c_{y,p}^s, \\
c_{o,p}^d &= \begin{cases} (p_n - \kappa)h_p^d - \bar{d} & \text{if the bubble bursts} \\ (p_p - \kappa)h_p^d - \bar{d} & \text{if the bubble persists} \end{cases},
\end{aligned}
$$

and savers' consumption and asset positions are given by:

$$
\begin{aligned}
a_p^d &= \bar{d}, \\
h_p^s &= \frac{e^s - \bar{d}\frac{1}{1-\pi} - c_{y,p}^s - \tilde{b}}{p_p}, \\
c_{y,p}^s &= \frac{1}{\beta(1-\pi)}, \\
c_{o,p}^s &= \begin{cases} (p_n - \kappa)h_p^s + \bar{d} & \text{if the bubble bursts} \\ (p_p - \kappa)h_p^s + \bar{d} + \tilde{b} & \text{if the bubble persists} \end{cases}.
\end{aligned}
$$

3. *Assume $1 + r_n < 1 - \pi$. If the initial bubble price satisfies $\tilde{b}_0 < \tilde{b}$, then there exists no asymptotic pure bubble equilibrium, as any equilibrium bubble must vanish: $\lim_{t\to\infty} \tilde{b}_t = 0$. If $\tilde{b}_0 > \tilde{b}$, then there exists no pure bubble equilibrium, as any equilibrium bubble must explode: $\lim_{t\to\infty} \tilde{b}_t = \infty$. If $\tilde{b}_0 = \tilde{b}$, then there exists a unique asymptotic pure bubble equilibrium, and this equilibrium coincides with the pure bubble steady state.*

Proof. Appendix A.5. □

5.2 Welfare analysis

Are the welfare implications of a pure bubble different from those of a housing bubble? As before, the welfare in the pure bubble steady sate is defined as $U_p^i = U(h_p^i, c_{y,p}^i, c_{o,p}^i)$, where h_p^i, $c_{y,p}^i$, and $c_{o,p}^s$ are given in lemma 6. In a low interest rate environment, a standard result is that a pure bubble allows savers to store their income into old age more efficiently and hence improves their welfare relative to the bubble-less benchmark. This result also holds in our environment:

Lemma 7. *The pure bubble improves welfare for savers: $U_p^s > U_n^s$.*

Proof. Appendix A.6 □

On the other hand, there are two effects of the pure bubble on borrowers. First, if e^s sufficiently large and $\bar{d}$ sufficiently small that the bubble does not only crowd out savers' young age consumption, then the bubble also crowds out savers' investment in housing for savings purposes. This reduces the housing price, making it easier for borrowers to obtain the housing asset. This crowd-out effect increases borrowers' welfare. Second, the bubble raises the interest rate that borrowers must pay on debt (from $1+r_n < 1-\pi$ to $1+r = 1-\pi$), reducing borrowers' purchasing power. This interest-rate effect reduces borrowers' welfare.

Whether the crowd-out effect dominates the interest-rate effect depends on the size of the bubble. A large bubble $\tilde{b}$ has a strong crowd-out effect, leading to a small housing price $p = p_p$ and an overall positive welfare effect on borrowers. The size of the bubble $\tilde{b}$, in turn, depends on the degree of credit friction in the economy $\bar{d}$ and the size of the savers' endowment, e^s. The following lemma formalizes this intuition:

Lemma 8. *The welfare effect of the pure bubble on borrowers is ambiguous. On the one hand, if the credit limit $\bar{d}$ is sufficiently small and savers' endowment e^s sufficiently large, then the pure bubble improves borrowers' welfare: $U_p^d > U_n^d$. On the other hand, if $e^s < \tilde{e}^s$, where $\tilde{e}^s \equiv \frac{\beta(\pi p_n - \kappa)(\frac{1}{1-\pi}(\bar{d}+\frac{1}{\beta})+\tilde{b}) - p_n\theta}{\beta(\pi p_n - \kappa)}$, then the pure bubble reduces borrowers' welfare.*

Proof. Appendix A.7. □

5.3 Welfare comparison across bubble steady states

We can also compare household welfare across bubble equilibria since both bubbles can exist under the same condition (that $1 + r_n < 1 - \pi$). We can show that borrowers have higher lifetime welfare in the pure bubble equilibrium than in the housing bubble equilibrium. This is because the housing bubble raises the interest rate and the price of housing for borrowers.

The pure bubble also raises the interest by the same amount, but does not raise the housing price. Therefore, borrowers are always better able to acquire housing (and consume when young) in a pure bubble equilibrium than in a housing bubble equilibrium. Furthermore, if the pure bubble is sufficiently large to crowd out savers' investment in housing for saving purposes, borrowers can acquire housing more easily in the pure bubble equilibrium than in the bubble-less equilibrium or housing bubble equilibrium.

Savers also prefer the pure bubble to the housing bubble. The housing bubble improves savers' welfare over the bubble-less equilibrium by raising the interest rate and increasing the housing price, but still requires savers to save excess wealth in housing. Since housing requires maintenance at cost κ, the efficiency of housing as a store of value is always inferior to that of the pure bubble. The pure bubble then allows savers to more efficiently allocate wealth across periods of life in the pure bubble equilibrium than housing does in the housing bubble equilibrium.

Corollary 9. *The pure bubble steady state Pareto dominates the housing bubble steady state.*

Proof. Appendix A.8 □

6   Conclusion

We have shown that a housing bubble, or, more generally, a bubble attached to a fundamentally useful asset, has heterogeneous welfare effects on borrowers and savers. By providing an additional investment vehicle, it raises the returns from investment for savers and thus improves their welfare. However, by raising the interest rate on debt and raising the housing price, the housing bubble negatively affects the welfare of borrowers, who need debt to finance their purchase of housing. Overall, our model implies a feedback loop on inequality: high income inequality leads to an environment with low interest rates, which facilitate housing bubbles, which in turn have regressive welfare effects.

Even though we model income inequality in an overly simple way, our model suggests an interesting possibility that, in an economy with sufficient income inequality and credit friction, consumption and welfare inequality can be exacerbated by the housing bubble. This result is relevant for the ongoing debate about inequality, especially in the U.S.. Future research can explore the interaction between bubbles and inequality in a more sophisticated framework of endogenous income/wealth inequality (e.g., with longer-lived overlapping generations).

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A Appendix: Proofs

A.1 Proof of lemma 1

Proof. First we show that all equilibrium values are time-invariant. From the first order conditions of savers and borrowers in steady state:

$$p_t = \frac{1}{1+r_t}\left(\frac{\theta}{\beta h_t^s} + p_{t+1} - \kappa\right),$$

$$p_t = \frac{\theta c_{t,y}^d}{h_t^d} + \beta c_{t,y}^d(p_{t+1} - \kappa).$$

We can then also define h_t^s, h_t^d, $c_{y,t}^s$, and $c_{y,t}^d$ as functions of r_t and p_t. Since no variable in time t is dependent upon $t-1$ (or any previous time period), it must be the case that no choices made by the young in time t influence the young in time $t+1$. Since r_t and p_t are the only time-variant variables in any of these expressions, it must be the case that, if r_t, p_t satisfy the first order conditions in one time period, they must satisfy the same condition in all time periods. Therefore, r_t, p_t are invariant and must be equal to the steady state values r_n,p_n for all $t > 0$. With the credit constraint binding for borrowers, the steady state consumption and housing for savers will be given as in the statement of the lemma.

Now, to prove the uniqueness of r_n and p_n we use the expressions from (6) and (7):

$$r_n = \frac{1}{p_n}\left(\frac{\theta}{\beta h_n^s} - \kappa\right),$$

$$p_n = \frac{c_{y,n}^d\left(\frac{\theta}{h_n^d} - \kappa\right)}{1 - \beta c_{y,n}^d}.$$

Combining (6) with the savers' steady state allocation of housing yields:

$$\frac{\beta r_n}{1+r_n}\left(\frac{1}{\beta} - \bar{d} - e^s(1+r_n)\right) = \frac{\kappa}{p_n},$$

where, taking p_n as given from (7) we see r_n is determined as the solution to a quadratic with one solution corresponding with positive young age consumption for both household types. Using this solution for r_n, we then arrive at a unique solution for p_n since $c_{y,n}^d$ is not a function of the price of housing. Therefore, r_n and p_n are unique and the prices and allocations given in the Lemma must represent a unique solution to the household's problem in P1. $\square$

A.2 Proof of Lemma 2

Part 1: Existence

Proof. First, we show that the housing bubble exists with $b > 0$ if and only if the bubble-less equilibrium interest rate is $1 + r_n < 1 - \pi$. To derive the bubble component, we look at the first order conditions of credit-constrained borrowers:

$$f_t + b_t = \frac{\theta c_{t,y}^d}{h_{t,y}^d} + [(1-\pi)(f_{t+1} + b_{t+1}) + \pi p_n - \kappa]\beta c_{t,y}^d, \tag{18}$$

21

which equates the gross price of housing to the discounted net lifetime utility dividends and resale value for credit constrained borrowers. Equation (11) and the steady state version of (18) give:

$$b = \frac{c_{y,h}^d \left(\frac{\theta}{h_h^d} - \kappa \right) - p_n(1 - \beta c_{y,h}^d)}{1 - \beta c_{d,h}^y(1 - \pi)},$$

which implies that $b > 0$ if and only if:

$$p_n < \frac{c_{y,h}^d \left(\frac{\theta}{h_h^d} - \kappa \right)}{1 - \beta c_{y,h}^d}.$$

But, from (7) we know that, in the bubble-less equilibrium:

$$p_n = \frac{\beta c_{y,n}^d \left(\frac{\theta}{h_n^d} - \kappa \right)}{1 - \beta c_{y,n}^d},$$

which implies:

$$\frac{c_{y,h}^d \left(\frac{\theta}{h_h^d} - \kappa \right)}{1 - \beta c_{y,h}^d} > \frac{\beta c_{y,n}^d \left(\frac{\theta}{h_n^d} - \kappa \right)}{1 - \beta c_{y,n}^d}. \tag{19}$$

Therefore, the bubble exists in equilibrium ($b > 0$) if and only if (19) holds.

Suppose $1 + r_n < 1 - \pi$ and $b = 0$. From the savers first order conditions we know that $c_y^s = \frac{1}{\beta(1+r)}$ in any equilibrium. Therefore, $1 + r_n < 1 - \pi$ implies $c_{y,h}^s < c_{y,n}^s$ since the interest rate in the housing bubble equilibrium must be $1 + r = 1 - \pi$. We also know from the cumulative young-age budget constraint that, for either equilibrium, the total amount of resources spent by young households must equal the total endowment e: $p_n + c_{y,h}^s + c_{y,h}^d = e = p_n + c_{y,n}^s + c_{y,n}^d$. Since p_n is unchanging and $c_{y,h}^s < c_{y,n}^s$, it must be that $c_{y,h}^d > c_{y,n}^d$. From the borrower's budget constraint, $1 + r_n < 1 - \pi$, and $c_{y,h}^d > c_{y,n}^d$ we know $h_h^d < h_n^d$. We therefore know that $1 + r_n < 1 - \pi$ implies $c_{y,h}^d > c_{y,n}^d$ and $h_h^d < h_n^d$. But, $c_{y,h}^d > c_{y,n}^d$ and $h_h^d < h_n^d$ means that (19) holds, which means that $b > 0$. Therefore $r_n < -\pi \Rightarrow b > 0$.

To prove that $b > 0$ implies $1 + r_n < 1 - \pi$, we start by assuming $b > 0$ and $1 + r_n \geq 1 - \pi$. First, $1 + r_n \geq 1 - \pi$ implies $c_{y,h}^s \geq c_{y,n}^s$ since $c_y^s = \frac{1}{\beta(1+r)}$ in any equilibrium. Second, we know that savers' housing is determined by the savers' young-age budget constraint. With $b > 0$, $1 + r_n \geq 1 - \pi = 1 + r_h$, and $c_{y,h}^s \geq c_{y,n}^s$, we know that savers must acquire *strictly* less housing in the housing bubble equilibrium. Since housing allocations must sum to one in any equilibrium, we then know that $h_h^d > h_n^d$. Finally, from the cumulative young-age budget constraint, $p_n + b + c_{y,h}^s + c_{y,h}^d = e = p_n + c_{y,n}^s + c_{y,n}^d$, we know $b > 0$ and $c_{y,h}^s \geq c_{y,n}^s$ imply $c_{y,h}^d < c_{y,n}^d$. Therefore, we have shown that, if $b > 0$ *and* $1 + r_n \geq 1 - \pi$, $c_{y,h}^d < c_{y,n}^d$ and $h_h^d > h_n^d$. But, this is a contradiction since $c_{y,h}^d < c_{y,n}^d$ and $h_h^d > h_n^d$ implies that (19) fails and $b = 0$. Therefore, we have shown $b > 0 \Rightarrow 1 + r_n < 1 - \pi$.

With these two results, we have shown $b > 0 \Leftrightarrow r_n < -\pi$, as desired. Note that it follows trivially that there can be no housing bubble equilibrium if $\pi \geq \kappa$. Intuitively, this result shows that there can only be a housing bubble in equilibrium if there is insufficient storage in the economy for savers to efficiently store wealth for old age. □

Part 2: Allocations and prices

Proof. With $b > 0$, $f = p_n$, and $1 + r_n < 1 - \pi$, housing allocations follow from the household budget constraints as given in the Lemma.

Borrowers are credit constrained while savers are not:

$$a^d = -\bar{d} = -a^s.$$

Borrowers' consumption and housing asset holdings are given by:

$$
\begin{aligned}
h_h^d &= \frac{e^d + \bar{d}\frac{1}{1-\pi} - c_{y,h}^d}{p_n + b}, \\
c_{y,h}^d &= e - (p_n + b) - c_{y,h}^s, \\
c_{o,h}^d &= \begin{cases} (p_n - \kappa)h_h^d - \bar{d} & \text{The bubble bursts} \\ (p_n + b - \kappa)h_h^d - \bar{d} & \text{The bubble persists} \end{cases}
\end{aligned}
$$

Savers' consumption and housing asset holdings are given by:

$$
\begin{aligned}
h_h^s &= \frac{e^s - \bar{d}\frac{1}{1-\pi} - c_{y,h}^s}{p_n + b}, \\
c_{y,h}^s &= \frac{1}{\beta(1-\pi)}, \\
c_{o,h}^s &= \begin{cases} (p_n - \kappa)h_h^s + \bar{d} & \text{The bubble bursts} \\ (p_n + b - \kappa)h_h^d + \bar{d} & \text{The bubble persists} \end{cases}
\end{aligned}
$$

Finally we need to prove that when the economy reverts to the bubble-less steady state after the bubble collapses. Suppose that the exists in a steady state in time $t - 1$ and bursts in time t. Then, the young households born in time t would face a lifetime optimization problem that is identical to that faced by households in the bubble-less equilibrium. Therefore, since the steady state is the unique solution for the household's problem in the bubble-less setup, it must be that households will choose an equilibrium with bubble-less equilibrium allocations with prices $p = p_n$ and $r = r_n$. $\square$

Part 3: Convergence

In any period t, the no arbitrage condition on the bubble asset requires that the bubble grows at the interest rate:

$$(1 - \pi)b_{t+1} = (1 + r_t)b_t.$$

This no arbitrage condition gives b_{t+1} as a function of r_t and b_t: $b_{t+1} = g(r_t, b_t)$. Note that $\frac{\partial g}{\partial r_t} = \frac{b_t}{1-\pi} \geq 0$, i.e., the bubble size in $t + 1$ is an increasing function in the interest rate r_t. Furthermore, savers' first order conditions yield:

$$r_t = \frac{1}{p_n}\left(\frac{\theta}{\beta h_t^s} - \kappa\right).$$

We have already shown that in any housing bubble equilibrium, the fundamental value, p_n, is independent of b_t, and, from the savers' young-age budget constraint we know that h_t^s is decreasing

23

in b_t, as well. Therefore, we can express the interest rate as a function of b_t: $r_t = f(b_t)$ with $\frac{\partial f}{b_t}$ given by:

$$\frac{\partial f}{\partial b_t} = -\frac{1}{p_t} \cdot \frac{\theta}{\beta(h_t^s)^2} \frac{\partial h_t^s}{\partial b_t} > 0. \tag{20}$$

This implies that the interest rate $r_t = f(b_t)$ is increasing in the size of the bubble b_t. Therefore the equilibrium dynamics can be characterized by the following equations, with b_t being the state variable:

$$
\begin{aligned}
r_t &= f(b_t), \\
b_{t+1} &= g(r_t, b_t),
\end{aligned}
$$

with f and g both monotonic in b_t and r_t, respectively. Note that $r_t = -\pi$ and $b_t = b$ is the unique steady state of this system. Now suppose that the initial bubble size is small: $b_0 < b$. Then, $r_0 = f(b_0) < f(b) = -\pi$. This means that, in period $t = 1$:

$$b_1 = g(r_0) = \frac{1 + r_0}{1 - \pi} b_0 < b_0.$$

This inequality implies that $r_1 = f(b_1) < r_0 < 0$, and $b_2 = g(r_1) = \frac{1+r_1}{1-\pi} b_1 < b_1$. By induction, we can prove that $\{b_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$ are decreasing sequences. Therefore, at any period t:

$$b_t = \left[\Pi_{s=0}^{t-1} \left(\frac{1 + r_s}{1 - \pi} \right) \right] b_0 \leq \left(\frac{1 + r_0}{1 - \pi} \right)^t b_0.$$

Since $\frac{1+r_0}{1-\pi} < 1$, it follows that $\lim_{t \to \infty} b_t \leq \lim_{t \to \infty} \left(\frac{1+r_0}{1-\pi} \right)^t b_0 = 0$. Thus, in any pure equilibrium with initial $b_0 < b$, it must be that the bubble vanishes: $\lim_{t \to} b_t = 0$.

Now, suppose $b_0 > b$. Then $r_0 = f(b_0) > f(b) = -\pi$ and

$$b_1 = g(r_o) = \frac{1 + r_0}{1 - \pi} b_0 > b_0,$$

Which means that $r_1 = f(b_1) > r_0$ and $b_2 = g(r_1) = \frac{1+r_1}{1-\pi} b_1 > b_1$. By induction, we can prove that $\{b_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$ are increasing sequences. Therefore, at any time t, b_t given by

$$b_t = \left[\Pi_{s=0}^{t-1} \left(\frac{1 + r_s}{1 - \pi} \right) \right] b_0 \geq \left(\frac{1 + r_0}{1 - \pi} \right)^t b_0.$$

Since $\frac{1+r_0}{1-\pi} > 1$, it follows that the bubble explodes to infinity: $\lim_{t \to \infty} b_t = \infty$. Therefore, the bubble economy converges to the bubble steady state only when the initial bubble is $b_0 = b$.

A.3 Proof of Lemma 3

Proof. Savers must have greater purchasing power when young in a steady state with a housing bubble than in a steady state without one since $1 + r_n < 1 - \pi$. Also, the optimal consumption for young savers is lower in a steady state with a housing bubble than in a steady state with no bubbles. This means $(p_n + b)h_h^s > p_n h_n^s$, and savers can consume more when old under a housing bubble than when there are no bubbles. Since savers are consuming less when young despite having more wealth *and* lending to borrowers, we know that saving for old age consumption is the challenge facing savers.

Formally, denote lifetime welfare for savers with net housing price and interest rate $p = p_n + b_h$ and $r = -\pi$ (as in the housing bubble equilibrium) as $U_h^s(h^s, c_y^s, c_o^s, a^s)$. Suppose savers choose to acquire the bubble-less equilibrium bundle when young: $h^s = h_n^s, c_y^s = c_{y,n}^s, a^s = \bar{d}$. This bundle is feasible under housing bubble equilibrium prices and leads to old age consumption $c_o^s = (p_n + b(1 - \pi) - \kappa)h_n^s + \bar{d}$. Comparing c_o^s with $c_{o,n}^s$, we see that $c_o^s - c_{o,n}^s = b(1 - \pi)h_n^s > 0$. Therefore, the bundle for the bubble-less equilibrium is feasible under the housing bubble and

$$U_h^s(h_h^s, c_{y,h}^s, c_{o,h}^s, \bar{d}) \geq U_h^s(h_n^s, c_{y,n}^s, c_o^s, \bar{d}) > U_n^s(h_n^s, c_{y,n}^s, c_{o,n}^s, \bar{d}),$$

where $U_n^s(h, c_y, c_o, a)$ is savers' welfare with the price and interest rate of the bubble-less equilibrium with bundle (h, c_y, c_o, a).

The housing bubble thus improves savers' lifetime utilities in the steady state by raising the interest rate on debt and making housing a more efficient store of value. Intuitively, savers are better able to consume and acquire housing when young due to the increased interest rate, while the bubble component of the housing price allows them to more efficiently store wealth for old age consumption. $\qquad \square$

A.4 Proof of Lemma 4

Proof. The total purchasing power of young borrowers in any equilibrium is given by $e^d + \bar{d}\frac{1}{1+r}$. Therefore, young borrowers have more purchasing power in a bubble-less steady state compared to a housing bubble steady state if and only if $e^d + \bar{d}\frac{1}{1+r_n} > e^d + \frac{\bar{d}}{1-\pi}$. This inequality is always satisfied if $1 + r_n < 1 - \pi$. Since borrowers have access to less wealth in the housing bubble steady state than in the bubble-less steady state, it must also be that: $c_{y,h}^d + (p_n + b)h_h^d < c_{y,n}^d + p_n h_n^d$.

Consider the two possibilities: $c_{y,h}^d \geq c_{y,n}^d$ or $c_{y,h}^d < c_{y,n}^d$. If $c_{y,h}^d < c_{y,n}^d$, the borrowers' fist order condition shows that, if $c_{y,h}^d < c_{y,n}^d$, then $h_h^d < h_n^d$. The only way borrowers could experience greater welfare in this case is if the bubble helps borrowers save for old-age consumption through their housing acquisition. But, if sacrificing young-age consumption and housing for old-age consumption is optimal for the borrower, then borrowers could take on less debt in the bubble-less steady state, acquire $h^d < h_n^d$, $c_y^d \leq c_{y,n}^d$, and consume more when old. Formally, there must exist $\epsilon > 0$ such that:

$$U_n^d\left(h_n^d - \frac{\epsilon}{(1+r_n)p_n}, c_y^d, , c_{o,n}^d + \epsilon, \bar{d} - \epsilon\right) > U_n^d(h_n^d, c_{y,n}^d, c_n^d, \bar{d}),$$

where $U_n^d(h, c_y, c_o, a)$ denotes the borrowers' welfare with price $p = p_n$ and interest rate $r = r_n$. This is impossible since the bundle $(h_n^d, c_{y,n}^d, c_{o,n}^d, \bar{d})$ is optimal for borrowers in the bubble-less equilibrium.

Similarly, if $c_{y,h}^d \geq c_{y,n}^d$, then $p_h h_h^d < p_n h_n^d$, which implies that $h_h^d < h_n^d$. If it is optimal to sacrifice housing (and, hence, old age consumption) for young age consumption in a bundle that is feasible in the bubble-less equilibrium, then there must exist $\epsilon > 0$ such that:

$$U_n^d\left(h_n^d - \frac{\epsilon}{p_n}, c_{y,n}^d + \epsilon, c_n^d - \epsilon, \bar{d}\right) > U_n^d(h_n^d, c_n^d, c_n^d, \bar{d}),$$

which is impossible since $(h_n^d, c_n^d, c_n^d, \bar{d})$ is the optimal bundle for borrowers in the bubble-less equilibrium. Therefore, the housing bubble *reduces* welfare for borrowers.

Intuitively, the bubble on housing introduces two difficulties for credit-constrained borrowers: first, the interest rate on debt increases, which makes borrowing to acquire housing and consume when young more expensive, second, the bubble increases the cost of acquiring housing. This means

that, when there is a housing bubble, borrowers must pay more over their lifetimes in debt to pay for housing at the inflated cost with the bubble. This leads to less lifetime consumption and less housing for borrowers. The bubble does make housing a more efficient store of value, but, if borrowers are already debt constrained, sacrificing young age consumption and housing for old age consumption cannot possibly improve their welfare.

It is important to remember that this unambiguously negative impact of the housing bubble on borrower welfare is brought about by its attachment to an object from which the households derive utility. If the bubble were attached to something that only yielded consumption in old age or was a store of value, as in the pure bubble case, borrowers could ignore the bubble asset completely, or would care only about the amount of wealth stored in the bubble asset instead of the amount of the bubble asset acquired. □

A.5 Proof of Lemma 6

Part 1: Existence

Proof. We will now show that the bubble exists with $\tilde{b} > 0$ if and only if $1 + r_n < 1 - \pi$. First, assume $1 + r_n < 1 - \pi$. We know from (17) that the bubble exists with $\tilde{b} > 0$ in the steady state if and only if:

$$e^s > \bar{d}\frac{1}{1-\pi} + c_{y,p}^s - \frac{\theta p_p}{\kappa - \pi\beta p_p}. \tag{21}$$

This implies that the pure bubble can exist only if savers have some endowment left over after lending, consuming the optimal amount in young age, and purchasing housing to satisfy utility purposes. Note, however, that we know from (15) in the bubble-less equilibrium that:

$$e = \bar{d}\frac{1}{1+r_n} + c_{y,n}^s - \frac{\theta p_n}{\kappa + r_n\beta p_n}, \tag{22}$$

which is the same condition as (21) only in the bubble-less equilibrium. Since the pure bubble can only increase the price of housing if savers decide to purchase *more* housing in the pure bubble equilibrium than the bubble-less steady state (which means that the bubble-less allocation is not actually optimal for savers and not an equilibrium), we can safely assume $p_p \leq p_n$. Then, with $1 + r_n < 1 - \pi$, $c_y^s = \frac{1}{\beta(1+r)}$, and (22), we know that:

$$e^s > \bar{d}\frac{1}{1-\pi} + c_{y,p}^s - \frac{\theta p_n}{\kappa - \pi\beta p_n},$$

which means that $\tilde{b} > 0$ even if the bubble fails to crowd out housing investment ($p_p = p_n$). Therefore, we have shown: $1 + r_n < 1 - \pi \Rightarrow \tilde{b} > 0$.

Now, suppose $\tilde{b} > 0$ and $1 + r_n \geq 1 - \pi$. Then, from (22) it *must* be the case that $p_p < p_n$. We also know that $c_{y,n}^s \leq c_{y,p}^s$. But, this means that savers find it optimal to drive interest rates down below r_n, purchase less housing, and consume more when young (relative to the bubble-less equilibrium). Since the bubble-less allocation is always feasible for savers (if savers choose $\tilde{b} = 0$), this must mean that there exists an allocation for savers $\{h^s < h_n^s, c_y^s > c_{y,n}^s, c_o^s \geq c_{0,n}^s, \bar{d}\}$ with equilibrium prices $p < p_n$ and $r < r_n$ that is feasible in the bubble-less equilibrium that yields strictly greater utility for savers than the bubble-less equilibrium bundle. This cannot be the case since we know the bubble-less equilibrium bundle to be optimal for unconstrained savers. Therefore: $\tilde{b} > 0 \Rightarrow 1 + r_n < 1 - \pi$. Thus, we have shown $\tilde{b} > 0 \Leftrightarrow 1 + r_n < 1 - \pi$. □

Part 2: Allocations and prices

Proof. First, we show that the debt constraint and short-selling constraint on the bubble asset must either both bind or both be nonbinding. First order conditions from the household's problem yield the following rule for debt and bubble acquisition:

$$\lambda_{d,t}^i \tilde{b} = \lambda_{b,t}^i,$$

where $\lambda_{d,t}$ and $\lambda_{b,t}$ are the Lagrange multipliers associated with the binding credit constraint and binding short-selling constraint on the bubble respectively and $i \in \{s, d\}$. Given $\tilde{b} > 0$, the debt constraint is non-binding if and only if the bubble asset short-selling constraint is non-binding. This means that a credit constrained household will never invest in the bubble asset, while a non-credit constrained household always will. Intuitively, the same household will never find it optimal both to save with the bubble and take on debt. Therefore, with p_p and $\tilde{b}$ as determined by (16) and (17) respectively, all allocations are as given in the Lemma. $\qquad\square$

Part 3: Convergence

Proof. In any period t, the no arbitrage condition on the bubble asset requires that the bubble grows at the interest rate:

$$(1 - \pi)\tilde{b}_{t+1} = (1 + r_t)\tilde{b}_t.$$

This no arbitrage condition gives $\tilde{b}_{t+1}$ as a function of r_t and $\tilde{b}_t$: $\tilde{b}_{t+1} = g(r_t, \tilde{b}_t)$. Note that $\frac{\partial g}{\partial r_t} = \frac{\tilde{b}_t}{1-\pi} \geq 0$, i.e., the bubble size in $t+1$ is an increasing function in the interest rate r_t. Furthermore, savers' first order conditions yield:

$$r_t = \frac{1}{p_t}\left(\frac{\theta}{\beta h_t^s} - \kappa\right).$$

We have already shown that any equilibrium p_t is decreasing in $\tilde{b}_t$, and, from the savers' young-age budget constraint we know that h_t^s is decreasing in $\tilde{b}_t$, as well. Therefore, we can express the interest rate as a function of $\tilde{b}_t$: $r_t = f(\tilde{b}_t)$ with $\frac{\partial f}{\partial \tilde{b}_t}$ given by:

$$\frac{\partial f}{\partial \tilde{b}_t} = \frac{-1}{p_t^2}\left(\frac{\theta}{\beta h_t^s} - \kappa\right)\frac{\partial p_t}{\tilde{b}_t} - \frac{1}{p_t}\cdot\frac{\theta}{\beta(h_t^s)^2}\frac{\partial h_t^s}{\partial \tilde{b}_t} > 0. \tag{23}$$

This implies that the interest rate $r_t = f(\tilde{b}_t)$ is increasing in the size of the bubble $\tilde{b}_t$. Therefore the equilibrium dynamics can be characterized by the following equations, with $\tilde{b}_t$ being the state variable:

$$\begin{aligned} r_t &= f(\tilde{b}_t), \\ \tilde{b}_{t+1} &= g(r_t, \tilde{b}_t), \end{aligned}$$

with f and g both monotonic in $\tilde{b}_t$ and r_t, respectively. Note that $r_t = -\pi$ and $\tilde{b}_t = \tilde{b}$ is the unique steady state of this system. Now suppose that the initial bubble size is small: $\tilde{b}_0 < \tilde{b}$. Then, $r_0 = f(\tilde{b}_0) < f(\tilde{b}) = -\pi$. This means that, in period $t = 1$:

$$\tilde{b}_1 = g(r_0) = \frac{1 + r_0}{1 - \pi}\tilde{b}_0 < \tilde{b}_0.$$

27

This inequality implies that $r_1 = f(\tilde{b}_1) < r_0 < 0$, and $\tilde{b}_2 = g(r_1) = \frac{1+r_1}{1-\pi}\tilde{b}_1 < \tilde{b}_1$. By induction, we can prove that $\{\tilde{b}_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$ are decreasing sequences. Therefore, at any period t:

$$\tilde{b}_t = \left[\Pi_{s=0}^{t-1}\left(\frac{1+r_s}{1-\pi}\right)\right]\tilde{b}_0 \leq \left(\frac{1+r_0}{1-\pi}\right)^t\tilde{b}_0.$$

Since $\frac{1+r_0}{1-\pi} < 1$, it follows that $\lim_{t\to\infty}\tilde{b}_t \leq \lim_{t\to\infty}\left(\frac{1+r_0}{1-\pi}\right)^t\tilde{b}_0 = 0$. Thus, in any pure equilibrium with initial $\tilde{b}_0 < \tilde{b}$, it must be that the bubble vanishes: $\lim_{t\to}\tilde{b}_t = 0$.

Now, suppose $\tilde{b}_0 > \tilde{b}$. Then $r_0 = f(\tilde{b}_0) > f(\tilde{b}) = -\pi$ and

$$\tilde{b}_1 = g(r_o) = \frac{1+r_0}{1-\pi}\tilde{b}_0 > \tilde{b}_0,$$

Which means that $r_1 = f(\tilde{b}_1) > r_0$ and $\tilde{b}_2 = g(r_1) = \frac{1+r_1}{1-\pi}\tilde{b}_1 > \tilde{b}_1$. By induction, we can prove that $\{\tilde{b}_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$ are increasing sequences. Therefore, at any time t, $\tilde{b}_t$ given by

$$\tilde{b}_t = \left[\Pi_{s=0}^{t-1}\left(\frac{1+r_s}{1-\pi}\right)\right]\tilde{b}_0 \geq \left(\frac{1+r_0}{1-\pi}\right)^t\tilde{b}_0.$$

Since $\frac{1+r_0}{1-\pi} > 1$, it follows that the bubble explodes to infinity: $\lim_{t\to\infty}\tilde{b}_t = \infty$. Therefore, the bubble economy converges to the bubble steady state only when the initial bubble is $\tilde{b}_0 = \tilde{b}$. □

A.6 Proof of Lemma 7

Proof. Since we assume e^s sufficiently large that $p_p < p_n$ and $r_n < -\pi$ we know that the bubble-less allocation is feasible but not optimal in the bubble-less equilibrium. Formally, we can define b' such that $0 < b' < \tilde{b}$. We can then define h' as the housing and $h_p^s < h'$, and let $c_y' = c_{y,n}^s$. With a bubble of size $b' > 0$, it must then be that $c_o' \geq c_{o,n}^s$ since $r_n < -\pi$. Therefore there exists a feasible bundle of housing under the pure bubble equilibrium that strictly dominates the optimal bundle for the bubble-less equilibrium and :

$$U(h_n^s, c_{y,n}^s, c_{o,n}^s) < U(h', c_{y,n}^s, c_o') \leq U(h_p^s, c_{y,p}^s, c_{o,p}^s).$$

□

A.7 Proof of Lemma 8

Proof. We show in this proof that the welfare effects of the pure bubble on borrowers is ambiguous and depends upon the parameterization of the model. To demonstrate this, consider this expression of (17) solving for the price of housing:

$$p_p = \frac{\kappa\beta(e^s - \bar{d}\frac{1}{1-\pi} - c_{y,p}^s - b)}{\beta\pi(e^s - \bar{d}\frac{1}{1-\pi} - c_{y,p}^s - \tilde{b}) + \theta} \tag{24}$$

We can see that p_p is decreasing in $\tilde{b}$, so if e^s (and, therefore, $\tilde{b}$) sufficiently large and $\bar{d}$ sufficiently small, the borrowers can afford to purchase more housing in the pure bubble equilibrium than in the bubble-less equilibrium *without* sacrificing young-age consumption: $h_p^d > h_n^d$, $c_{y,p}^d \geq c_{y,n}^d$. Consider parameters $e^s = e_1^s$, $\bar{d} = \bar{d}_1$ such that this is the case. Since the decline in p_p is non-linear with $\tilde{b}$,

the effect on $c_{o,p}^d$ is also ambiguous. However, since we assume parameters such that the borrowers are credit constrained, we know that the borrower is willing to sacrifice old age consumption for housing and consumption when young even at the margin. Therefore, with e^s (and, consequently, $\tilde{b}$) sufficiently large and $\bar{d}$ sufficiently small the pure bubble improves welfare for borrowers over the bubble-less equilibrium.

Now consider $e_2^s < e_1^s$ and $\bar{d}_2 > \bar{d}_1$ such that the bubble exists in equilibrium but *does not* decrease the price of housing. In this case, the bubble would only crowd out savers' young age consumption and not their housing purchase. Formally, with $r_{n,2}$ and $p_{n,2}$ as the bubble-less equilibrium interest rate and housing price with $e^s = e_2^s$ and $\bar{d} = \bar{d}_2$:

$$p_{n,2} = \frac{\kappa\beta(e_2^s - \bar{d}_2\frac{1}{1-\pi} - c_{y,p}^s - b_2)}{\beta\pi(e_2^s - \bar{d}_2\frac{1}{1-\pi} - c_{y,p}^s - \tilde{b}_2) + \theta},$$

and we can see that borrowers will experience lower lifetime welfare in the pure bubble equilibrium than in the bubble-less equilibrium in this case since the bubble is raising the interest rate on debt while failing to crowd out savers' investment in housing. Therefore, borrowers face higher borrowing costs and no corresponding decrease in housing prices that could improve their lifetime welfare.

Formally, for any fixed $\bar{d}$, define $\tilde{e}^s$ as the maximum endowment of savers that allows for both the bubble to exist with $\tilde{b} > 0$ *and* $p_p = p_n$:

$$\tilde{e}^s = \frac{\beta(\pi p_n - \kappa)(\frac{1}{1-\pi}(\bar{d} + \frac{1}{\beta}) + \tilde{b}) - p_n\theta}{\beta(\pi p_n - \kappa)}.$$

Then, for any e^s for which $\tilde{b} > 0$ *and* $e^s \le \tilde{e}^s$, the pure bubble will harm borrowers' welfare (relative to the bubble-less equilibrium).

But, there must also exist $\hat{e}^s > \tilde{e}^s$ such that the price and bubble price resulting from $\hat{e}^s$ $(\hat{p}_p, \hat{b})$ allow for $c_{y,p}^d \ge c_{y,n}^d$ and $h_p^d > h_n^d$. Specifically, we define $\hat{p}_p = p_n - \frac{1}{\beta}\left(\frac{1}{1+r_n} - \frac{1}{1-\pi}\right)$, then $\hat{e}^s$ and $\hat{b}$ are defined by:

$$\hat{e}^s = \frac{\beta(\pi\hat{p}_p - \kappa)(\frac{1}{1-\pi}(\bar{d} + \frac{1}{\beta}) + \hat{b}) - \hat{p}_p\theta}{\beta(\pi\hat{p}_p - \kappa)}.$$

Then, for $e^s \ge \hat{e}^s$, the pure bubble improves borrowers' welfare. $\qquad\square$

A.8 Proof of Corollary 9

Proof. Part 1: Comparison of savers' welfare

Since both bubble equilibria must have the same interest rate, the preferable equilibrium for savers will be the equilibrium that allows savers to most efficiently save for old age consumption (conditional upon the bubble remaining intact). We already know that $c_{y,h}^s = c_{y,p}^s$ since $1+r = 1-\pi$ in both the pure bubble and housing bubble equilibria. Then, from the savers' young age budget constraint in either bubble setup:

$$p_p h_p^s + \tilde{b} = (p_n + b)h_h^s,$$

since $c_{y,h}^s = c_{y,p}^s$ and lending is the same in either equilibrium ($\bar{d}\frac{1}{1-\pi}$). The bubble equilibrium with the greatest consumption in old age is therefore the equilibrium in which savers can most efficiently save. The pure bubble is then preferable to the housing bubble:

$$
\begin{aligned}
c_{o,p}^s > c_{o,h}^s \Leftrightarrow \tilde{b} &> (p_n + b - \kappa)h_h^s - (p_p - \kappa)h_p^s \\
\tilde{b} &> \tilde{b} + p_p h_p^s - \kappa h_h^s - p_p h_p^s + \kappa h_p^s \\
c_{o,p}^s > c_{o,h}^s \Leftrightarrow h_h^s &> h_p^s
\end{aligned}
$$

Therefore, as long as $h_h^s > h_p^s$, savers have greater utility in the pure bubble equilibrium than in the housing bubble equilibrium. But, we know that $h_h^s > h_p^s$ must hold in *any* setup where both bubbles can exist since $h_p^s \geq h_h^s$ implies $p_p \geq p_n + b$, which is infeasible with a pure bubble of positive value. Therefore $U_p^s > U_h^s$.

Part 2: Comparison of borrowers' welfare

If $\bar{d}$ sufficiently small that $h_p^s > h_n^s$ and $U_p^d > U_n^d$, it immediately follows that, since we have already shown that the housing bubble reduces welfare for borrowers compared to the bubble-less equilibrium that:

$$U_h^d < U_n^d < U_p^d.$$

Now, suppose that e^s and $\bar{d}$ are *not* such that borrowers are better off in the pure bubble than the bubble-less equilibrium. In any equilibrium with credit constrained borrowers, welfare of borrowers is determined by how much housing and consumption borrowers are able to acquire when young. Since borrowers can take on the same amount of debt in either bubble equilibrium ($1 + r = 1 - \pi$ in both equilibria), we know that:

$$p_p h_d^s + c_{y,p}^d = (p_n + b) h_h^s + c_{y,p}^d,$$

where b is, again, the bubble value in the housing bubble steady state. From this equation we can see that the housing bubble bundle for young borrowers is feasible but non-optimal in the pure bubble equilibrium since we know that, for $b_h > 0$, $p_p < p_n + b_h$. Therefore, borrowers experience higher welfare under the pure bubble than in a housing bubble and $U_p^d \geq U_h^d$. □